at the base of the scutellum; one similar spot, oblong and transverse on the disc of each hemelytrum: body beneath yellow with a black bronzed spot on each side of the mesostethium; a narrow band of the same colour at the base of the venter, and a row of five similar spots on each side: the abdominal point reaches only the insertion of the intermediate feet (*C. incarnatus*, Am. & Serv.). Long, 25—30 mill.

Var. *b.* :—Large; head with antennæ deep black; pronotum orange, with the anterior margin deep black: scutellum orange, immaculate: hemelytra orange with a median fuscous spot: wings fuscous: margin of abdomen variegated with orange and black: feet deep black (*C. aurantius*, Fabr.). Long, 25-30 mill.

Var. *c.* :—Scutellum, hemelytra and pectus immaculate. Ceylon.

Reported from Corea, Japan, Java, Sumatra, Borneo, Siam, Malacca, Singapore, Tenasserim, Ceylon, Madras, Bombay, Bengal, Pondicherry, Silhat, Assam. The Indian Museum has specimens from Tenasserim, Assam, Sikkim, Calcutta, Karachi, Malabar. Varies in colour from a sordid yellow, to orange and a bright maroon red, with and without the black spots.

II.—*A General Theorem on the Differential Equations of Trajectories.*

—*By* Asutosh Mukhopadhyay, M. A., F. R. A. S., F. R. S. E.

[Received November 17th;—Read December 7th, 1887.]

Contents.

§. 1. *Introduction.*

In a paper on "The Differential Equation of a Trajectory," which was read at the last May meeting of the Society, (*Journal*, 1887, Vol. LVI, Part. II, pp. 117—120; *Proceedings*, 1887, p. 151), I pointed out that Mainardi's complicated solution (reproduced by Boole) of the problem of determining the oblique trajectory of a system of confocal ellipses, was equivalent to a pair of remarkably simple equations which admitted of an interesting geometrical interpretation. Believing, as I firmly did, that every simple mathematical result could be established by a correspondingly simple process, I naturally thought it worth while to re-examine the whole question, to see if the very artificial process of

Mainardi, by no means less complicated than his result, could be materially simplified. I was, thus, led to the following very general theorem, which it is my object in the present paper to establish and illustrate, and, which shews that whenever the coordinates of any point on a curve can be expressed by means of a single variable parameter, the coordinates of the corresponding point on the trajectory may be similarly expressed; and, as an immediate corollary to my theorem, I have pointed out the relation which connects it with the theory of Conjugate Functions.*

§. 2. *Theorem.*

Theorem.—If the coordinates of any point on a curve are expressed by means of a variable parameter θ, by the two equations

$$x = f_1\,(\theta, a),$$
$$y = f_2\,(\theta, b),$$

where a and b are two arbitrary constants; and, if we seek the oblique trajectory of the system of curves obtained by varying a and b, subject to any condition which can be analytically represented by means of a parameter ψ, as equivalent to the system

$$a = \mathrm{F}_1\,(\psi, h),$$
$$b = \mathrm{F}_2\,(\psi, h),$$

where h is a known constant; the coordinates of the corresponding point on the trajectory are given by the system

$$\mathrm{X} = f_1 \left\{\theta, \mathrm{F}_1\,(\psi, h)\right\},$$
$$\mathrm{Y} = f_2 \left\{\theta, \mathrm{F}_2\,(\psi, h)\right\},$$

where ψ is given as a function of θ by the differential equation

$$\frac{d\psi}{d\theta} = \frac{n\,\mathrm{L}}{\mathrm{N} - n\,\mathrm{M}}$$

where $n = \tan\alpha,$

α being the angle of intersection of the curve and the trajectory, and

$$\mathrm{L} = \left(\frac{df_1}{d\theta}\right)^2 + \left(\frac{df_2}{d\theta}\right)^2$$
$$\mathrm{M} = \frac{df_1}{d\theta}\,\frac{df_1}{d\psi} + \frac{df_2}{d\theta}\,\frac{df_2}{d\psi}$$
$$\mathrm{N} = \frac{df_2}{d\theta}\,\frac{df_1}{d\psi} - \frac{df_1}{d\theta}\,\frac{df_2}{d\psi}.$$

To establish this theorem, let us first fix the ideas by confining our attention to one definite member of the given family of curves as well as to one of the trajectories; then it is clear that the common point of intersection of the curve and the trajectory, may be arbitrarily regarded as a

* For a full analysis of this paper, see the *Proceedings* for 1887, pp. 250-251.

point, either on the one, or on the other; and, from each point of view, the coordinates satisfy two entirely different equations, though their actual values are the same in both cases; hence, if the coordinates of the point, regarded as a point on the curve, be furnished by the system

$$x = f_1(\theta, a), \qquad \ldots\ldots\ldots\ldots \quad (1)$$
$$y = f_2(\theta, b), \qquad \ldots\ldots\ldots\ldots \quad (2)$$

and the trajectory is obtained by varying a and b subject to the limitations

$$a = F_1(\psi, h), \qquad \ldots\ldots\ldots\ldots \quad (3)$$
$$b = F_2(\psi, h), \qquad \ldots\ldots\ldots\ldots \quad (4)$$

the coordinates of the corresponding point on the trajectory must be obtained by substituting in (1) and (2) the values of a and b from (3) and (4), *viz*, we have

$$X = f_1\left\{\theta, F_1(\psi, h)\right\} \qquad \ldots\ldots\ldots\ldots \quad (5)$$
$$Y = f_2\left\{\theta, F_2(\psi, h)\right\} \qquad \ldots\ldots\ldots\ldots \quad (6)$$

In the next place, we have to determine ψ as a function of θ, and this is easily obtained from the condition that the trajectory intersects the curve at a constant angle α. Now, it is well-known that

$$\frac{dy}{dx}, \frac{dY}{dX}$$

are the trigonometrical tangents of the angles which the tangents to the curve and to the trajectory, at their common point of intersection, make with the axis of x; hence, if $n = \tan\alpha$, we have

$$n = \frac{\dfrac{dy}{dx} - \dfrac{dY}{dX}}{1 + \dfrac{dy}{dx}\dfrac{dY}{dX}}$$
$$= \frac{\dfrac{dy}{d\theta}\dfrac{dX}{d\theta} - \dfrac{dx}{d\theta}\dfrac{dY}{d\theta}}{\dfrac{dx}{d\theta}\dfrac{dX}{d\theta} + \dfrac{dy}{d\theta}\dfrac{dY}{d\theta}}. \qquad \ldots\ldots\ldots\ldots \quad (7)$$

Remembering that in differentiating X and Y with respect to θ, we must regard θ as a function of ψ, but not so in the case of x and y, we have

$$\frac{dx}{d\theta} = \frac{df_1}{d\theta}, \qquad \frac{dy}{d\theta} = \frac{df_2}{d\theta}$$
$$\frac{dX}{d\theta} = \frac{df_1}{d\theta} + \frac{df_1}{d\psi}\frac{d\psi}{d\theta},$$
$$\frac{dY}{d\theta} = \frac{df_2}{d\theta} + \frac{df_2}{d\psi}\frac{d\psi}{d\theta},$$

which lead to the values

$$\frac{dy}{d\theta}\frac{dX}{d\theta}-\frac{dx}{d\theta}\frac{dY}{d\theta}$$

$$=\frac{d\psi}{d\theta}\left\{\frac{df_1}{d\psi}\frac{df_2}{d\theta}-\frac{df_1}{d\theta}\frac{df_2}{d\psi}\right\}$$

$$\frac{dx}{d\theta}\frac{dX}{d\theta}+\frac{dy}{d\theta}\frac{dY}{d\theta}$$

$$=\left(\frac{df_1}{d\theta}\right)^2+\left(\frac{df_2}{d\theta}\right)^2+\frac{d\psi}{d\theta}\left\{\frac{df_1}{d\theta}\frac{df_1}{d\psi}+\frac{df_2}{d\theta}\frac{df_2}{d\psi}\right\}.$$

Hence, putting

$$\mathrm{L}=\left(\frac{df_1}{d\theta}\right)^2+\left(\frac{df_2}{d\theta}\right)^2, \qquad \ldots\ldots\ldots\ldots \quad (8)$$

$$\mathrm{M}=\frac{df_1}{d\theta}\frac{df_1}{d\psi}+\frac{df_2}{d\theta}\frac{df_2}{d\psi}, \qquad \ldots\ldots\ldots\ldots \quad (9)$$

$$\mathrm{N}=\frac{df_2}{d\theta}\frac{df_1}{d\psi}-\frac{df_1}{d\theta}\frac{df_2}{d\psi}, \qquad \ldots\ldots\ldots\ldots \quad (10)$$

we have finally, from (7), the equation

$$\frac{d\psi}{d\theta}=\frac{n\,\mathrm{L}}{\mathrm{N}-n\,\mathrm{M}}, \qquad \ldots\ldots\ldots\ldots \quad (11)$$

which is exactly the theorem enunciated above.

It may not be altogether unprofitable to note that the trajectory is determined by two conditions, *viz.*, in the first place, we have to vary the constants in a definite manner; and, in the second place, the trajectory is to intersect the curve at a given angle; the first of these conditions leads to the actual values of the coordinates of any point on the trajectory, furnished by (5) and (6), while the second condition determines the relation between θ and ψ which enter into the values of those coordinates.

§. 3. *Application to Mainardi's Problem.*

Example I.—In order to test the power and generality of this theorem, we shall apply it to solve Mainardi's problem of determining the oblique trajectory of a system of confocal ellipses. The primitive ellipse being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \ldots\ldots\ldots\ldots \quad (12)$$

we get the confocal system by varying a and b subject to the condition

$$a^2-b^2=h^2. \qquad \ldots\ldots\ldots\ldots \quad (13)$$

The coordinates of any point on the ellipse are given by

$$x=a\cos\theta,$$
$$y=b\sin\theta,$$

while the relation between a and b given in (13), is equivalent to

$$a = h \cosh \psi,$$
$$b = h \sinh \psi,$$

so that, the coordinates of any point on the trajectory are given by

$$X = h \cos \theta \cosh \psi, \qquad \ldots\ldots\ldots\ldots \quad (14)$$
$$Y = h \sin \theta \sinh \psi. \qquad \ldots\ldots\ldots \quad (15)$$

Again, to determine the relation between θ and ψ, we have

$$f_1 = h \cos \theta \cosh \psi,$$
$$f_2 = h \sin \theta \sinh \psi,$$

which lead to the system

$$\frac{df_1}{d\theta} = -h \sin \theta \cosh \psi,$$
$$\frac{df_2}{d\theta} = h \cos \theta \sinh \psi,$$
$$\frac{df_1}{d\psi} = h \cos \theta \sinh \psi,$$
$$\frac{df_2}{d\psi} = h \sin \theta \cosh \psi,$$

and, these give

$$L = h^2 (\sin^2 \theta \cosh^2 \psi + \cos^2 \theta \sinh^2 \psi),$$
$$M = 0,$$
$$N = h^2 (\sin^2 \theta \cosh^2 \psi + \cos^2 \theta \sinh^2 \psi),$$

so that, the differential equation (11) becomes

$$\frac{d\psi}{d\theta} = n$$

whence, $$\psi = n (\lambda + \theta),$$

where λ is the constant of integration. Substituting in (14) and (15), we see finally that the coordinates of any point on the oblique trajectory of a system of confocal ellipses, are given by

$$X = h \cos \theta \cosh n (\lambda + \theta),$$
$$Y = h \sin \theta \sinh n (\lambda + \theta),$$

which is exactly the system of equations to which Mainardi's result was reduced in my former paper, and geometrically interpreted there.

§. 4. *Other applications of the Theorem.*

Example II.—To find the oblique trajectory of the system of confocal hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $$a^2 + b^2 = h^2.$$

The coordinates of any point on the hyperbola are given by

$$x = a \cosh \theta,$$
$$y = b \sinh \theta,$$

where

$$a = h \cos \psi,$$
$$b = h \sin \psi,$$

so that the coordinates of any point on the trajectory are given by

$$X = h \cosh \theta \cos \psi,$$
$$Y = h \sinh \theta \sin \psi.$$

To determine ψ as a function of θ, we have

$$f_1 = h \cosh \theta \cos \psi,$$
$$f_2 = h \sinh \theta \sin \psi,$$

whence

$$\frac{df_1}{d\theta} = h \sinh \theta \cos \psi,$$
$$\frac{df_2}{d\theta} = h \cosh \theta \sin \psi,$$
$$\frac{df_1}{d\psi} = -h \cosh \theta \sin \psi,$$
$$\frac{df_2}{d\psi} = h \sinh \theta \cos \psi,$$

and, therefore,

$$L = h^2 \left\{ \sinh^2 \theta \cos^2 \psi + \cosh^2 \theta \sin^2 \psi \right\}$$
$$M = 0$$
$$N = -h^2 \left\{ \sinh^2 \theta \cos^2 \psi + \cosh^2 \theta \sin^2 \psi \right\}$$

The differential equation (11) becomes

$$\frac{d\psi}{d\theta} = -n,$$

so that

$$\psi = n\,(\lambda - \theta),$$

where, of course, λ is a constant different from the λ in the solution of Mainardi's problem. The coordinates of any point on the oblique trajectory of a system of confocal hyperbolas are, therefore, given by

$$X = h \cosh \theta.\ \cos n\,(\lambda - \theta).$$
$$Y = h \sinh \theta.\ \sin n\,(\lambda - \theta).$$

If we put

$$\theta = \lambda - \frac{\phi}{n}, \quad \lambda\, n = -\mu,$$

these equations may be written

$$X = \quad h \cos \phi.\ \cosh \frac{1}{n}\,(\mu + \phi),$$
$$Y = -h \sin \phi.\ \sinh \frac{1}{n}\,(\mu + \phi),$$

which system is slightly different from what has been obtained above as the solution of Mainardi's problem; but the equations are obviously capable of a geometrical interpretation closely analogous to what is given in my former paper.

If we had to obtain by the ordinary method the oblique trajectory of a system of confocal hyperbolas, we should have to eliminate a and b from the equations

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ a^2+b^2=h^2,$$

$$\frac{dy}{dx}=p=-\frac{\frac{x}{a^2}+\frac{ny}{b^2}}{\frac{nx}{a^2}-\frac{y}{b^2}}.$$

The result may be expressed in the form

$$\left\{(nx-y)+(x-ny)\,p\right\}\left\{(x+ny)+(nx+y)\,p\right\}$$
$$=h^2\,(n+p)(1+np).$$

But, it is surely no agreeable task to have to find the actual equation of the trajectory by integrating this differential equation.

Assuming the expressions for the coordinates of any point on the oblique trajectory of a system of confocal ellipses, it is easy to write down the expressions for the coordinates of any point on the oblique trajectory of a system of confocal hyperbolas. Consider the point of intersection of an ellipse and its trajectory, and draw through this point the confocal hyperbola; then, since the ellipse and hyperbola cut each other orthogonally, the trajectory, which intersects the ellipse at an angle α, will intersect the hyperbola at an angle $\left(\frac{\pi}{2}+\alpha\right)$, in both cases measuring the angle of intersection in the same sense; the trajectory, therefore, is also the oblique trajectory of the confocal hyperbolas (at an angle $\frac{\pi}{2}+\alpha$), and the coordinates of any point on it, as such, will, therefore, be obtained by writing for n ($=\tan\alpha$), $-\frac{1}{n}\left(=\tan\left[\frac{\pi}{2}+\alpha\right]\right)$

Example III.—To find the oblique trajectory of a system of parabolas which have a common principal axis and which touch each other at their common vertex, and, the equations of which are, accordingly, obtained by varying a in

$$y^2=4ax.$$

The coordinates of any point on the curve are given by

$$x=a\tan^2\theta,$$
$$y=2a\tan\theta.$$

As the two constants of the general theorem are here equal, the coordinates of any point on the trajectory are given by

$$X = \psi \tan^2 \theta$$
$$Y = 2 \psi \tan \theta.$$

To determine ψ as a function of θ, we have

$$f_1 = \psi \tan^2 \theta$$
$$f_2 = 2 \psi \tan \theta,$$

which give

$$\frac{df_1}{d\theta} = 2 \psi \tan \theta \sec^2 \theta,$$
$$\frac{df_2}{d\theta} = 2 \psi \sec^2 \theta,$$
$$\frac{df_1}{d\psi} = \tan^2 \theta$$
$$\frac{df_2}{d\psi} = 2 \tan \theta,$$

so that we have

$$L = 4 \psi^2 \sec^6 \theta$$
$$M = 2 \psi \tan \theta \sec^2 \theta \; (2 + \tan^2 \theta)$$
$$N = -2 \psi \tan^2 \theta \sec^2 \theta$$

and the differential equation for ψ becomes

$$\frac{d\psi}{d\theta} = -\frac{2n \; \psi \sec^4 \theta}{\tan \theta \; (2n + \tan \theta + n \tan^2 \theta)}.$$

This may be written

$$\frac{d\psi}{\psi} = -\frac{2n \sec^4 \theta \; d \theta}{\tan \theta \; (2n + \tan \theta + n \tan^2 \theta)},$$

which, by putting $\tan \theta = z$, reduces to

$$\frac{d\psi}{\psi} = -2n \; \frac{(1+z^2) \; dz}{z \; (2n+z+nz^2)}$$

or,

$$\frac{d\psi}{\psi} = \frac{3}{2} \frac{dz}{2n+z+nz^2} - \frac{dz}{z} - \frac{1}{2} \frac{(2nz+1) \; dz}{2n+z+nz^2}.$$

Integrating, we have

$$\log \frac{\psi}{\lambda} = \frac{3}{2\sqrt{1-8n^2}} \log \frac{2nz+1-\sqrt{1-8n^2}}{2nz+1+\sqrt{1-8n^2}}$$
$$- \log z - \tfrac{1}{2} \log \; (2n+z+nz^2),$$

which gives

$$\psi = \frac{\lambda}{\tan \theta \sqrt{(2n + \tan \theta + n \tan^2 \theta)}} \left\{ \frac{2n \tan \theta + 1 - \sqrt{1-8n^2}}{2n \tan \theta + 1 + \sqrt{1-8n^2}} \right\}^{\frac{3}{2\sqrt{1-8n^2}}}.$$

This holds so long as $8n^2 \angle 1$, or, if α be the angle of the trajectory

$$\tan\alpha \angle \frac{1}{2\sqrt{2}}.$$

If $\tan\alpha$ be greater than this value, the corresponding value of ψ will be still more complex, but may easily be found. In the particular case where

$$\tan\alpha = \frac{1}{2\sqrt{2}},$$

the differential equation for ψ reduces to

$$\frac{d\psi}{\psi} = 3\sqrt{2}\,\frac{dz}{(z+\sqrt{2})^2} - \frac{dz}{z} - \frac{dz}{z+\sqrt{2}}$$

Integrating and substituting for z, we have

$$\psi\tan\theta\,(\tan\theta+\sqrt{2}) = e^{\frac{-3\sqrt{2}}{\tan\theta+\sqrt{2}}}.$$

If the orthogonal trajectory be required, the expression for ψ admits of considerable simplification, for, then we have $n = \infty$, and the differential equation for ψ becomes

$$\frac{d\psi}{\psi} = -\frac{dz}{z} - \tfrac{1}{2}\,\frac{z\,dz}{1+\frac{1}{2}z^2},$$

which on integration leads to

$$\log\frac{\psi}{\lambda} = -\log z - \tfrac{1}{2}\log\left(1+\tfrac{1}{2}z^2\right),$$

or,

$$\psi z\left(1+\tfrac{1}{2}z^2\right)^{\frac{1}{2}} = \lambda,$$

which, by putting $z = \tan\theta$, reduces to

$$\psi^2 = \frac{2\lambda^2}{\tan^2\theta\,(2+\tan^2\theta)}.$$

The coordinates, therefore, of any point on the trajectory are given by

$$X^2 = \psi^2\tan^4\theta = \frac{2\lambda^2\tan^2\theta}{2+\tan^2\theta},$$

$$Y^2 = 4\psi^2\tan^2\theta = \frac{8\lambda^2}{2+\tan^2\theta},$$

which easily shew that the trajectory is the ellipse

$$y^2 + 2x^2 = 4\lambda^2.$$

Example IV.—To obtain the oblique trajectory of a pencil of coplanar rays radiating from a point, and whose equation is, therefore, obtained by varying a in

$$y = ax.$$

First Method.

The coordinates of any point on the line are given by

$$y = a\theta,$$
$$x = \theta,$$

so that the coordinates of any point on the trajectory are

$$X=\theta,$$
$$Y=\psi\theta,$$

where, to determine ψ as a function of θ, we have

$$f_1=\theta,$$
$$f_2=\psi\theta,$$

which furnish the system

$$\frac{df_1}{d\theta}=1,\ \frac{df_2}{d\theta}=\psi,$$
$$\frac{df_1}{d\psi}=0,\ \frac{df_2}{d\psi}=\theta,$$

and by virtue of these, we have

$$L=1+\psi^2,$$
$$M=\theta\,\psi,$$
$$N=-\theta,$$

whence, the differential equation for ψ is

$$\frac{d\psi}{d\theta}=\frac{n\,L}{N-n\,M}=\frac{n\,(1+\psi^2)}{-\theta-n\theta\,\psi}$$

which gives

$$\frac{1+n\,\psi}{1+\psi^2}\,d\psi=-n\,\frac{d\theta}{\theta}.$$

Integrating, we get

$$\tan^{-1}\psi+\frac{n}{2}\log\,(1+\psi^2)=-n\log\frac{\theta}{\lambda},$$

which easily reduces to

$$\theta=\frac{\lambda}{\sqrt{1+\psi^2}}\,e^{-\frac{1}{n}\tan^{-1}\psi}$$

Hence, finally, the coordinates of any point on the trajectory are given by

$$X=\frac{\lambda}{\sqrt{1+\psi^2}}\,e^{-\frac{1}{n}\tan^{-1}\psi}$$

$$Y=\frac{\lambda\,\psi}{\sqrt{1+\psi^2}}\,e^{-\frac{1}{n}\tan^{-1}\psi}$$

It is not difficult to shew that these values lead to a well-known result; for we have

$$\frac{Y}{X}=\psi$$

and $$(X^2+Y^2)^{\frac{1}{2}}=\lambda\,e^{-\frac{1}{n}\tan^{-1}\psi}$$

Transforming to polar coordinates, by putting

$$X = r\cos\phi,\ Y = r\sin\phi,$$

we have

$$\tan\phi = \psi$$

$$r = \lambda\, e^{-\frac{1}{n}\tan^{-1}\psi},$$

whence,

$$r = \lambda\, e^{-\frac{\phi}{n}},$$

which is the logarithmic spiral.

Second method.

We might also have proceeded as follows, *viz.*, putting $a = \tan\beta$, the coordinates of any point on the line are given by

$$x = e^{\theta}\cos\beta,$$
$$y = e^{\theta}\sin\beta.$$

The coordinates of any point on the trajectory are, therefore, given by

$$X = e^{\theta}\cos\psi,$$
$$Y = e^{\theta}\sin\psi.$$

To determine ψ as a function of θ, we have

$$f_1 = e^{\theta}\cos\psi,$$
$$f_2 = e^{\theta}\sin\psi,$$

whence, we have the system

$$\frac{df_1}{d\theta} = e^{\theta}\cos\psi,$$
$$\frac{df_2}{d\theta} = e^{\theta}\sin\psi,$$
$$\frac{df_1}{d\psi} = -e^{\theta}\sin\psi,$$
$$\frac{df_2}{d\psi} = e^{\theta}\cos\psi,$$

which furnish us with the values

$$L = e^{2\theta},\ M = 0,\ N = e^{-2\theta}.$$

The differential equation for ψ becomes

$$\frac{d\psi}{d\theta} = -n,$$

whence $\qquad \psi = n(\lambda - \theta).$

The coordinates of any point on the trajectory are, consequently, given by

$$X = e^{\theta}\cos n(\lambda - \theta),$$
$$Y = e^{\theta}\sin n(\lambda - \theta),$$

and it is not difficult to shew that these values belong to the logarithmic spiral.

Example V.—To find the oblique trajectory of a system of circles which touch a given straight line at a given point, and whose equation is, therefore, obtained by varying r in

$$x^2+y^2=2rx.$$

The coordinates of any point on the circle are given by

$$x=r\ (1+\cos\theta),$$
$$y=r\sin\theta,$$

so that, the coordinates of any point on the trajectory are given by

$$X=\psi\ (1+\cos\theta),$$
$$Y=\psi\sin\theta.$$

To determine ψ as a function of θ, we have

$$f_1=\psi\ (1+\cos\theta),$$
$$f_2=\psi\sin\theta,$$

which lead to the system

$$\frac{df_1}{d\theta}=-\psi\sin\theta$$
$$\frac{df_2}{d\theta}=\psi\cos\theta$$
$$\frac{df_1}{d\psi}=1+\cos\theta$$
$$\frac{df_2}{d\psi}=\sin\theta,$$

whence, we have

$$L=\psi^2$$
$$M=-\psi\sin\theta$$
$$N=\psi\ (1+\cos\theta).$$

The differential equation for ψ reduces to

$$\frac{d\psi}{d\theta}=\frac{n\,\psi}{1+\cos\theta+n\sin\theta}.$$

Writing $n=\tan\alpha$, where α is the angle of the trajectory, we have

$$\frac{d\psi}{\psi}=\sin\alpha\ \frac{d\ (\theta-\alpha)}{\cos\alpha+\cos\ (\theta-\alpha)}.$$

Integrating, we have at once

$$\log\frac{\psi}{\lambda}=\log\frac{\cos\dfrac{\alpha}{2}+\sin\dfrac{\alpha}{2}\tan\dfrac{\theta-\alpha}{2}}{\cos\dfrac{\alpha}{2}-\sin\dfrac{\alpha}{2}\tan\dfrac{\theta-\alpha}{2}},$$

whence

$$\psi = \lambda \frac{\cos\left(\frac{\theta}{2} - a\right)}{\cos\frac{\theta}{2}}.$$

The equations

$$X = \psi\ (1+\cos\theta) = 2\ \psi \cos^2\frac{\theta}{2}$$

$$Y = \psi \sin\theta = 2\ \psi \sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

which give the coordinates of any point on the trajectory, therefore, become

$$X = 2\lambda\cos\frac{\theta}{2}\cos\left(a - \frac{\theta}{2}\right)$$

$$Y = 2\lambda\sin\frac{\theta}{2}\cos\left(a - \frac{\theta}{2}\right).$$

Since

$$X^2 + Y^2 = 4\lambda^2\cos^2\left(a - \frac{\theta}{2}\right)$$

it is easily shewn that the trajectory is the circle

$$x^2 + y^2 = 2\lambda\ (x\cos a + y\sin a).$$

Example VI.—To find the oblique trajectory of a system of parabolas which have a common focus and principal axis, and whose equation is, therefore, obtained by varying m in

$$y^2 = 4m\ (x+m).$$

Putting $$m = a^2,$$

any point on the curve is seen to be given by

$$x = \theta^2 - a^2,$$

$$y = 2a\theta.$$

The coordinates of any point on the trajectory are, therefore, given by

$$X = \theta^2 - \psi^2,$$

$$Y = 2\ \theta\ \psi,$$

where ψ is to be determined as a function of θ from the system

$$f_1 = \theta^2 - \psi^2$$

$$f_2 = 2\theta\psi,$$

so that we have

$$\frac{df_1}{d\theta} = 2\theta,\quad \frac{df_2}{d\theta} = 2\psi,$$

$$\frac{df_1}{d\psi} = -2\psi,\quad \frac{df_2}{d\psi} = 2\theta,$$

and these values shew that

$$L = 4\ (\theta^2 + \psi^2)$$
$$M = 0$$
$$N = -4\ (\theta^2 + \psi^2).$$

The differential equation for ψ, consequently, becomes

$$\frac{d\psi}{d\theta} = -n$$

whence

$$\psi = n\ (\lambda - \theta).$$

Hence, finally, the coordinates of any point on the trajectory are given by

$$X = \theta^2 - n^2\ (\lambda - \theta)^2.$$
$$Y = 2n\theta\ (\lambda - \theta).$$

Since X and Y are two quadratic functions of the parameter θ, it is clear that the trajectory must be a conic; in fact, the actual equation is

$$(1 + n^2)^2\ (x^2 + y^2) = \left\{(n^2 - 1)\ x + 2ny - 2n^2\lambda^2\right\}^2,$$

which may be thrown into the form

$$\left\{2nx - (n^2 - 1)\ y\right\}^2 = 4n^2\lambda^2\left\{n^2\lambda^2 - (n^2 - 1)\ x - 2ny\right\},$$

which shews that the trajectory is a parabola, and, if $n = \tan\alpha$, the polar equation is

$$\sqrt{r}.\ \sin\left(\alpha + \frac{\phi}{2}\right) = \lambda \sin\alpha.$$

Example VII.—To find the oblique trajectory of the system of curves obtained by varying b in the equation

$$e^x \sin y = ab.$$

The coordinates of any point on the curve are given by

$$x = \log a\sqrt{\theta^2 + b^2}$$
$$y = \tan^{-1}\frac{b}{\theta}.$$

The coordinates of any point on the trajectory are, therefore, given by

$$X = \log a\sqrt{\theta^2 + \psi^2}$$
$$Y = \tan^{-1}\frac{\psi}{\theta}.$$

To determine ψ as a function of θ, we have

$$f_1 = \log a + \tfrac{1}{2}\log\ (\theta^2 + \psi^2),$$
$$f_2 = \tan^{-1}\frac{\psi}{\theta},$$

which give the values

$$\frac{df_1}{d\theta} = \frac{\theta}{\theta^2 + \psi^2}$$

$$\frac{df_2}{d\theta}=\frac{-\psi}{\theta^2+\psi^2}$$
$$\frac{df_1}{d\psi}=\frac{\psi}{\theta^2+\psi^2}$$
$$\frac{df_2}{d\psi}=\frac{\theta^2}{\theta^2+\psi^2},$$

whence

$$L=1,\ M=0,\ N=-1,$$

and the differential equation for ψ is

$$\frac{d\psi}{d\theta}=-n,$$

which gives

$$\psi=n(\lambda-\theta).$$

The coordinates of any point on the trajectory are, therefore, given by

$$X=\log a\sqrt{\theta^2+n^2(\lambda-\theta)^2}$$
$$Y=\tan^{-1}\frac{n(\lambda-\theta)}{\theta}.$$

It can easily be shewn from this system that the actual equation of the trajectory is

$$e^x(\sin y+n\cos y)=a\lambda n,$$

or, if α be the angle of the trajectory, this becomes

$$e^x\sin(y+\alpha)=a\lambda\sin\alpha.$$

§. 5. *Conjugate Functions.*

It will be remarked that in some of the examples given above, the integration of the differential equation for ψ was materially facilitated whenever we found

$$M=0,\ L=\pm N.$$

It is, therefore, a matter of importance to discover under what circumstances this may be expected to happen.

Theorem.—The coordinates of any point on a curve being given by

$$x=f_1(\theta,a),$$
$$y=f_2(\theta,b),$$

and, the coordinates of the corresponding point on the trajectory by

$$X=f_1\left\{\theta,F_1(\psi,h)\right\},$$
$$Y=f_2\left\{\theta,F_2(\psi,h)\right\},$$

if we have

$$\psi=n(\lambda+\theta),$$

and

$$M\equiv\frac{df_1}{d\theta}\frac{df_1}{d\psi}+\frac{df_2}{d\theta}\frac{df_2}{d\psi}=0,$$

to prove that f_1 and f_2 must be conjugate functions of ψ and θ.

To establish this, we see that the conditions given, *viz.*,

$$\psi = n(\lambda + \theta),\ \mathrm{M} = 0,$$

reduce the differential equation

$$\frac{d\psi}{d\theta} = \frac{n\,\mathrm{L}}{\mathrm{N} - n\,\mathrm{M}}$$

to the condition

$$\mathrm{L} = \mathrm{N}.$$

Now, since $\mathrm{M} = 0,$

we have

$$\frac{\dfrac{df_1}{d\theta}}{\dfrac{df_2}{d\theta}} = -\frac{\dfrac{df_2}{d\psi}}{\dfrac{df_1}{d\psi}} = \xi, \text{ say.}$$

Substituting in the value for N, we get

$$\mathrm{N} \equiv \frac{df_2}{d\theta}\frac{df_1}{d\psi} - \frac{df_1}{d\theta}\frac{df_2}{d\psi}$$

$$= (1+\xi^2)\frac{df_1}{d\psi}\frac{df_2}{d\theta}$$

$$= \frac{\dfrac{df_1}{d\psi}}{\dfrac{df_2}{d\theta}}\,\mathrm{L},$$

and, since $\mathrm{N} = \mathrm{L},$

we must have

$$\frac{df_1}{d\psi} = \frac{df_2}{d\theta}. \qquad \text{............ (16)}$$

Therefore

$$\mathrm{N} = \left(\frac{df_2}{d\theta}\right)^2 - \frac{df_1}{d\theta}\frac{df_2}{d\psi}$$

$$\mathrm{L} = \left(\frac{df_1}{d\theta}\right)^2 + \left(\frac{df_2}{d\theta}\right)^2,$$

whence

$$\frac{df_1}{d\theta} = -\frac{df_2}{d\psi} \qquad \text{............ (17)}$$

The two equations marked (16) and (17) make it manifest that f_1 and f_2 must be conjugate functions of ψ and θ.

In Mainardi's problem, which is the first example given above, we have $\psi = n(\lambda + \theta),\ \mathrm{M} = 0,$

so that the quantities

$$h\cos\theta\cosh\psi,\quad h\sin\theta\sinh\psi$$

are conjugate functions of ψ and θ; hence, we infer from a well-known property of these functions that the two curves

$$\cos x \cosh y = a$$
$$\sin x \sinh y = b$$

intersect orthogonally at every common point of intersection.

It may similarly be shewn that if we have

$$\psi = n(\lambda - \theta), \quad \mathrm{M} = 0,$$

the functions f_1 and f_2 are conjugate with respect to θ and ψ; for the above investigation remains unaltered, except in that we have

$$\mathrm{L} = -\mathrm{N},$$

so that (16) becomes

$$\frac{df_1}{d\psi} = -\frac{df_2}{d\theta}, \qquad \ldots\ldots\ldots\ldots (18)$$

and we have

$$\mathrm{N} = -\left(\frac{df_2}{d\theta}\right)^2 - \frac{df_1}{d\theta}\frac{df_2}{d\psi},$$
$$\mathrm{L} = \left(\frac{df_1}{d\theta}\right)^2 + \left(\frac{df_2}{d\theta}\right)^2,$$

whence

$$\frac{df_1}{d\theta} = \frac{df_2}{d\psi}, \qquad \ldots\ldots\ldots\ldots (19)$$

and, by virtue of (18) and (19), it is again manifest that f_1 and f_2 are two conjugate functions of θ and ψ. Consequently, as in the second example given above, we have

$$\psi = n(\lambda - \theta), \quad \mathrm{M} = 0,$$

the quantities

$$h \cosh \theta \cos \psi, \quad h \sinh \theta \sin \psi$$

are two conjugate functions of θ and ψ, and, the curves

$$\cosh x \cos y = a$$
$$\sinh x \sin y = b$$

are orthogonal trajectories of each other.

Again, it is an elementary principle in the theory of conjugate functions that if ϕ and ψ are any two conjugate functions of x and y; and if ξ, η are any two other conjugate functions of x and y: then, by putting ξ and η instead of x and y in the values of ϕ and ψ, we get two new conjugate functions of x and y. But, we have found above two pairs of such functions, *viz.*,

$$\left.\begin{aligned} \phi &= \sin x \sinh y \\ \psi &= \cos x \cosh y \end{aligned}\right\}$$
$$\left.\begin{aligned} \xi &= \cosh x \cos y \\ \eta &= \sinh x \sin y \end{aligned}\right\}.$$

Hence we have the two new conjugate functions

$$\sin\left\{\cosh x\cos y\right\}\sinh\left\{\sinh x\sin y\right\},$$

$$\cos\left\{\cosh x\cos y\right\}\cosh\left\{\sinh x\sin y\right\}.$$

We have, therefore, the theorem that the two transcendental curves

$$\sin\left\{\cosh x\cos y\right\}\sinh\left\{\sinh x\sin y\right\}=a$$

$$\cos\left\{\cosh x\cos y\right\}\cosh\left\{\sinh x\sin y\right\}=b$$

are orthogonal trajectories of each other. In the same manner, it may be shewn that the quantities which furnish the coordinates of any point on the trajectory in terms of θ and ψ, in the second method of establishing Example IV, as well as in Examples VI and VII, are conjugate functions.

We shall now give some examples in which the properties of conjugate functions will materially simplify the calculation.

Example VIII.—Consider the tricircular sextic

$$(x^2+y^2)(x^2+y^2+k^2)^2=a^2\left\{x^2(x^2+y^2-k^2)^2+y^2(x^2+y^2+k^2)^2\right\},$$

and suppose that its oblique trajectory is required when a is made to vary. Writing

$$a^2=1+b^2,$$

the equation may easily be thrown into the form

$$x^2(x^2+y^2+k^2)^2=a^2x^2(x^2+y^2-k^2)^2+b^2y^2(x^2+y^2+k^2)^2,$$

whence it can be shewn without much difficulty that the coordinates of any point on the sextic curve are given by the system

$$\frac{x^2}{k^2}=\frac{a-\cos\theta}{a+\cos\theta}\cdot\frac{b^2}{b^2+\sin^2\theta},$$

$$\frac{y^2}{k^2}=\frac{a-\cos\theta}{a+\cos\theta}\cdot\frac{\sin^2\theta}{b^2+\sin^2\theta},$$

and we seek the oblique trajectory, when a and b are made to vary subject to the conditions

$$a=\cosh\psi,$$

$$b=\sinh\psi.$$

The coordinates of any point on the trajectory are given by

$$\frac{X^2}{k^2}=\frac{\cosh\psi-\cos\theta}{\cosh\psi+\cos\theta}\cdot\frac{\sinh^2\psi}{\sinh^2\psi+\sin^2\theta}$$

$$\frac{Y^2}{k^2}=\frac{\cosh\psi-\cos\theta}{\cosh\psi+\cos\theta}\cdot\frac{\sin^2\theta}{\sinh^2\psi+\sin^2\theta}$$

To determine ψ as a function of θ, we have

$$X=f_1,\quad Y=f_2,$$

and then by actually calculating the values of

$$\frac{df_1}{d\theta}, \ \frac{df_2}{d\theta}, \ \frac{df_1}{d\psi}, \ \frac{df_2}{d\psi},$$

we can shew that

$$L+N=0, \quad M=0,$$

whence the differential equation for ψ becomes

$$\frac{d\psi}{d\theta}=-n$$

and

$$\psi=n\,(\lambda-\theta).$$

But, from the theorem we have established at the beginning of this section, we know that the same conclusions may be legitimately drawn without direct calculation, if we can prove f_1 and f_2 to be two conjugate functions, and we proceed to do so. Now we know that if

$$\tan\tfrac{1}{2}\,(\theta+\sqrt{-1}\,\psi)=A+\sqrt{-1}\,B,$$

the two conjugate functions A and B are given by the system

$$\frac{A^2}{B^2}=\frac{\sin^2\theta}{\sinh^2\psi}$$

$$A^2+B^2=\frac{\cosh\psi-\cos\theta}{\cosh\psi+\cos\theta},$$

whence it follows that

$$A^2=\frac{\cosh\psi-\cos\theta}{\cosh\psi+\cos\theta}\cdot\frac{\sin^2\theta}{\sinh^2\psi+\sin^2\theta}$$

$$B^2=\frac{\cosh\psi-\cos\theta}{\cosh\psi+\cos\theta}\cdot\frac{\sinh^2\psi}{\sinh^2\psi+\sin^2\theta}.$$

But these are the quantities which when multiplied by k^2 reproduce the squares of what we have called f_1 and f_2 above, which was to be proved. Hence we finally infer that the coordinates of any point on the sextic

$$(x^2+y^2)(x^2+y^2+k^2)^2=a^2\left\{x^2(x^2+y^2-k^2)^2+y^2(x^2+y^2+k^2)^2\right\}$$

may be represented by the equations

$$\frac{x^2}{k^2}=\frac{a-\cos\theta}{a+\cos\theta}\cdot\frac{b^2}{b^2+\sin^2\theta},$$

$$\frac{y^2}{k^2}=\frac{a-\cos\theta}{a+\cos\theta}\cdot\frac{\sin^2\theta}{b^2+\sin^2\theta},$$

where

$$a^2-b^2=1,$$

and, accordingly, the coordinates of any point on its oblique trajectory are furnished by the system

$$\frac{X^2}{k^2}=\frac{\cosh n(\lambda-\theta)-\cos\theta}{\cosh n(\lambda-\theta)+\cos\theta}\cdot\frac{\sinh^2 n(\lambda-\theta)}{\sinh^2 n(\lambda-\theta)+\sin^2\theta}.$$

$$\frac{Y^2}{k^2}=\frac{\cosh n(\lambda-\theta)-\cos\theta}{\cosh n(\lambda-\theta)+\cos\theta}\cdot\frac{\sin^2\theta}{\sinh^2 n(\lambda-\theta)+\sin^2\theta}.$$

Example IX.—Take, again, the curve

$$(x^2+y^2)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

and suppose that its oblique trajectory is required, when a and b are made to vary subject to the condition

$$a^2-b^2=h^2.$$

The coordinates of any point on the curve may be written

$$x=\frac{a\cos\theta}{a^2\cos^2\theta+b^2\sin^2\theta},$$

$$y=\frac{b\sin\theta}{a^2\cos^2\theta+b^2\sin^2\theta},$$

and we have also

$$a=h\cosh\psi$$

$$b=h\sinh\psi.$$

The coordinates of any point on the trajectory are, therefore, given by

$$\mathrm{X}=\frac{\cos\theta\cosh\psi}{h(\cos^2\theta\cosh^2\psi+\sin^2\theta\sinh^2\psi)}$$

$$=\frac{2\cos\theta\cosh\psi}{h(\cosh 2\psi+\cos 2\theta)}$$

$$\mathrm{Y}=\frac{\sin\theta\sinh\psi}{h(\cos^2\theta\cosh^2\psi+\sin^2\theta\sinh^2\psi)}$$

$$=\frac{2\sin\theta\sinh\psi}{h(\cosh 2\psi+\cos 2\theta)}.$$

To determine ψ as a function of θ, we have

$$f_1=\mathrm{X},\quad f_2=\mathrm{Y}.$$

But f_1 and f_2 are two conjugate functions; for we know that if we separate the real and imaginary parts of

$$\sec(\alpha+\sqrt{-1}\,\beta)=\mathrm{A}+\sqrt{-1}\,\mathrm{B},$$

we have

$$\mathrm{A}=\frac{2\cos\alpha\cosh\beta}{\cosh 2\beta+\cos 2\alpha}$$

$$\mathrm{B}=\frac{2\sin\alpha\sinh\beta}{\cosh 2\beta+\cos 2\alpha}.$$

Hence, by the theorem of this section, we have

$$\mathrm{L}+\mathrm{N}=0,\quad \mathrm{M}=0,$$

and the differential equation for ψ becomes

$$\frac{d\psi}{d\theta}=-n$$

whence $$\psi=n(\lambda-\theta).$$

We see, therefore, finally that the coordinates of any point on the oblique trajectory of the bicircular quartic

$$(x^2+y^2)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2},$$

which is obviously the inverse of an ellipse, may be represented by the system

$$X=\frac{2\cos\theta\cosh n(\lambda-\theta)}{h\left\{\cosh 2n(\lambda-\theta)+\cos 2\theta\right\}}$$

$$Y=\frac{2\sin\theta\sinh n(\lambda-\theta)}{h\left\{\cosh 2n(\lambda-\theta)+\cos 2\theta\right\}}$$

when a and b vary subject to the relation

$$h^2=a^2-b^2.$$

Example X.—Again, if we seek the oblique trajectory of the transcendental curve

$$\tan^2 y=\frac{b^2}{a^2}\,\frac{a^2e^{-x}-h^2e^{x}}{h^2e^{x}-b^2e^{-x}},$$

when a and b vary subject to the condition

$$a^2-b^2=h^2,$$

we see that the coordinates of any point on the curve are given by the system

$$2e^{2x}=\frac{a^2+b^2}{h^2}-\cos 2\theta.$$

$$\cot y=\frac{a}{b}\tan\theta.$$

But as

$$a=h\cosh\psi,$$
$$b=h\sinh\psi,$$

the coordinates of any point on the trajectory are given by

$$X=\tfrac{1}{2}\log\frac{\cosh 2\psi-\cos 2\theta}{2}$$

$$\cot Y=\coth\psi\tan\theta.$$

To determine ψ as a function of θ, we notice that the quantities

$$f_1=\tfrac{1}{2}\log 2\,(\cosh 2\psi-\cos 2\theta)-\log 2$$

$$f_2=\cot^{-1}(\coth\psi\tan\theta)$$

are two conjugate functions, being in fact exactly the two quantities which we obtain in separating the real and imaginary parts of

$$\log\sin(\theta+\sqrt{-1}\,\psi)$$

Hence, by the theorem of this section, we have

$$L+N=0,\ M=0,$$

and, as before,

$$\psi = n(\lambda - \theta).$$

Therefore, we finally infer that the coordinates of any point on the oblique trajectory of the curve

$$\tan^2 y = \frac{b^2}{a^2} \cdot \frac{a^2 e^{-x} - h^2 e^{x}}{h^2 e^{x} - b^2 e^{-x}},$$

when a and b vary subject to the relation

$$a^2 - b^2 = h^2,$$

are given by the system

$$2e^{2x} = \cosh 2n(\lambda - \theta) - \cos 2\theta.$$

$$\tan y = \tanh n(\lambda - \theta).\ \cot \theta.$$

From the theorem established in this section, it is again evident that, if

$$f_1(\theta, \psi),\quad f_2(\theta, \psi)$$

be any two conjugate functions of θ and ψ, and the equation of a curve be obtained by eliminating θ from the system

$$x = f_1(\theta, a)$$
$$y = f_2(\theta, b),$$

the equation of the oblique trajectory of this curve when a is made to vary is obtained by eliminating θ from the system

$$\mathrm{X} = f_1\left\{\theta,\ n(\lambda - \theta)\right\}$$
$$\mathrm{Y} = f_2\left\{\theta,\ n(\lambda - \theta)\right\}.$$

Similarly, if the equation of a curve is obtained by eliminating ψ from the system

$$x = f_1(a, \psi),$$
$$y = f_2(a, \psi),$$

the equation of the oblique trajectory of this curve when a is made to vary is obtained by eliminating ψ from the system

$$\mathrm{X} = f_1\left\{n(\lambda + \psi),\ \psi\right\}$$
$$\mathrm{Y} = f_2\left\{n(\lambda + \psi),\ \psi\right\}.$$

Again as from the well known formula for expanding

$$f(\theta + \sqrt{-1}\,\psi)$$

and separating its real and imaginary parts, *viz.*,

$$f_1 \equiv f(\theta) - \frac{\psi^2}{\underline{/2}} f''(\theta) + \frac{\psi^4}{\underline{/4}} f^{\mathrm{IV}}\theta - \&\text{c.},$$

$$f_2 \equiv \psi f'(\theta) - \frac{\psi^3}{\underline{/3}} f'''(\theta) + \&\text{c.},$$

we can determine at pleasure an infinite number of pairs of conjugate functions, it is clear that we may obtain without any difficulty an infinite number of curves whose oblique trajectories may be determined with ease by the theorems and methods of this paper; but it is needless to multiply instances, as the examples given above will, it is hoped, amply illustrate these observations.

16*th November* 1887.

Additional Note on Mainardi's Problem.

Since the above paper was read, I have been informed by Prof. Booth that Prof. Michael Roberts, in his Lectures on Differential Equations delivered at the University of Dublin, used to solve Mainardi's problem by the help of elliptic coordinates; I have not the opportunity of examining the solution arrived at by Prof. Roberts (as I believe it has never been published), but I give below the results I have obtained by means of the coordinates suggested.

If a be the semi-axis-major of the primitive conic, and h half the distance between its foci, its equation is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-h^2}=1,$$

and any member of the confocal family is obtained by varying a; so that, if λ, μ be the elliptic coordinates of any point P on the trajectory, they are determined from the system

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-h^2}=1,$$

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-h^2}=1,$$

viz., λ is the semi-axis-major of the ellipse, and μ the semi-axis-transverse of the hyperbola through P confocal to the primitive one; hence, solving between these equations, we have

$$x^2=\frac{\lambda^2\mu^2}{h^2},$$

$$y^2=\frac{-(\lambda^2-h^2)(\mu^2-h^2)}{h^2}.$$

Taking the logarithmic differential, we see that the element of arc of any curve through P is

$$ds^2=dx^2+dy^2=\frac{\lambda^2-\mu^2}{\lambda^2-h^2}\,d\lambda^2-\frac{\lambda^2-\mu^2}{\mu^2-h^2}\,d\mu^2.$$

Hence, if ds_1, ds_2 be the elements of arc of the confocal ellipse and hyperbola whose semiaxes are λ, μ, and which intersect orthogonally at P, we have, for the ellipse regarding λ as constant,

$$ds_1{}^2=\frac{\lambda^2-\mu^2}{h^2-\mu^2}\,d\mu^2,$$

and, for the hyperbola regarding μ as constant,

$$ds_2^2 = \frac{\lambda^2 - \mu^2}{\lambda^2 - h^2}\, d\lambda^2.$$

Now, if α be the angle of the trajectory, *viz.*, the angle at P between the trajectory and the ellipse (λ), we have clearly

$$\frac{ds_2}{ds_1} = \tan\alpha = n.$$

Hence

$$\frac{\lambda^2 - \mu^2}{\lambda^2 - h^2}\, d\lambda^2 = n^2 . \frac{\lambda^2 - \mu^2}{h^2 - \mu^2}\, d\mu^2,$$

or

$$\frac{d\lambda}{\sqrt{\lambda^2 - h^2}} = n\, \frac{d\mu}{\sqrt{h^2 - \mu^2}}.$$

Integrating, we have

$$\log\left(\lambda + \sqrt{\lambda^2 - h^2}\right) = - n \cos^{-1}\frac{\mu}{h} + k,$$

which is, accordingly, the equation of the trajectory in elliptic coordinates. It will be remarked that, though the application of elliptic coordinates removes the difficulties of integration, the result is not obtained in an appreciably simpler form; and, besides, the method is not one of general application, as it requires a knowledge of the elements of arc, as well of the given curve as of its orthogonal trajectory; the methods and theorems of this paper, however, effectually remove these disadvantages.

It may usefully be noted that if we use the inverse hyperbolic functions, the integral of

$$\frac{d\lambda}{\sqrt{\lambda^2 - h^2}} = n\, \frac{d\mu}{\sqrt{h^2 - \mu^2}}$$

may be written

$$\cosh^{-1}\frac{\lambda}{h} + n\cos^{-1}\frac{\mu}{h} = k,$$

and this at once shews that if we have

$$\lambda = h \cosh\theta,$$

where θ is a variable parameter, we must have

$$\mu = h \cos\frac{1}{n}(k - \theta).$$

In this form it is not difficult to identify our solution with Mainardi's result, *viz.*,

$$-2n\tan^{-1}\sqrt{\frac{h^2}{x\mathrm{M}} - 1} + \log\frac{1 - \sqrt{1 - \frac{\mathrm{M}}{x}}}{1 + \sqrt{1 - \frac{\mathrm{M}}{x}}} = \mathrm{C},$$

where M satisfies the quadratic

$$(x^2+y^2+h^2)\,\mathrm{M}=x\,(\mathrm{M}^2+h^2).$$

For since

$$x^2=\frac{\lambda^2\mu^2}{h^2}$$

$$y^2=\frac{-(\lambda^2-h^2)(\mu^2-h^2)}{h^2},$$

we have

$$x^2+y^2+h^2=\lambda^2+\mu^2,$$

whence the quadratic for M becomes

$$\mathrm{M}\,(\lambda^2+\mu^2)=\frac{\lambda\mu}{h}\,(\mathrm{M}^2+h^2),$$

which may be written

$$\mathrm{M}^2-h\left(\frac{\lambda}{\mu}+\frac{\mu}{\lambda}\right)\mathrm{M}+h^2=0,$$

the roots of which are

$$\mathrm{M}=\frac{h\mu}{\lambda},\ \frac{h\lambda}{\mu}.$$

Taking for the present

$$\mathrm{M}=\frac{h\mu}{\lambda},$$

we have

$$\mathrm{M}x=\mu^2,$$

$$\frac{\mathrm{M}}{x}=\frac{h^2}{\lambda^2}.$$

The equation of the trajectory, therefore, on substituting these values, becomes

$$-2n\tan^{-1}\sqrt{\frac{h^2}{\mu^2}-1}+\log\frac{1-\sqrt{1-\dfrac{h^2}{\lambda^2}}}{1+\sqrt{1-\dfrac{h^2}{\lambda^2}}}=\mathrm{C}.$$

Putting

$$h=\mu\sec\phi,$$

$$\mathrm{C}=2np,$$

where p is a new constant, this becomes

$$\frac{1-\sqrt{1-\dfrac{h^2}{\lambda^2}}}{1+\sqrt{1-\dfrac{h^2}{\lambda^2}}}=e^{2n(p+\phi)},$$

or

$$\frac{1}{\sqrt{1-\dfrac{h^2}{\lambda^2}}}=\frac{1+e^{2n(p+\phi)}}{1-e^{2n(p+\phi)}},$$

or

$$\frac{h^2}{\lambda^2} = 1 - \left\{\frac{1 - e^{2n(p+\phi)}}{1 + e^{2n(p+\phi)}}\right\}^2$$

$$= \frac{4e^{2n(p+\phi)}}{\left\{1 + e^{2n(p+\phi)}\right\}^2},$$

whence

$$\lambda = h.\ \frac{1 + e^{2n(p+\phi)}}{2e^{n(p+\phi)}}$$

$$= \frac{h}{2}\left\{e^{n(p+\phi)} + e^{-n(p+\phi)}\right\}$$

$$= h \cosh n(p+\phi).$$

We have, therefore, the system

$$\lambda = h \cosh n(p+\phi)$$

$$\mu = h \cos \phi.$$

If we put

$$n(p+\phi) = \theta,$$

$$\phi = \frac{\theta}{n} - p,$$

this is equivalent to the system obtained above, *viz.*,

$$\lambda = h \cosh \theta$$

$$\mu = h \cos \frac{1}{n}(\theta - pn).$$

If we had used for M the value

$$M = \frac{h\lambda}{\mu},$$

we should have to put

$$Mx = \lambda^2,$$

$$\frac{M}{x} = \frac{h^2}{\mu^2},$$

which shews that λ, μ would be interchanged in the equation of the trajectory, *viz.*, that would give the system

$$\lambda = h \cos \phi,$$

$$\mu = h \cosh n(p+\phi),$$

and it is important to notice that *this second system does not admit of being derived from the differential equation in elliptic coordinates,*

$$\frac{d\lambda}{\sqrt{\lambda^2 - h^2}} = n\frac{d\mu}{\sqrt{h^2 - \mu^2}}.$$

For the above system is the solution of the differential equation

$$n\frac{d\lambda}{\sqrt{h^2 - \lambda^2}} = \frac{d\mu}{\sqrt{\mu^2 - h^2}},$$

which is different from the one given above, *viz.*, this leads to the primitive

$$n \cos^{-1} \frac{\lambda}{h} + \cosh^{-1} \frac{\mu}{h} + k = 0,$$

so that, if

$$\lambda = h \cos \phi,$$

we must have

$$\mu = h \cosh n(p+\phi).$$

We see, then, that, because M is given by a quadratic, Mainardi's result is really equivalent to two, *viz.*, we have the two systems

$$\left.\begin{aligned} \lambda &= h \cosh n(p+\phi) \\ \mu &= h \cos \phi. \end{aligned}\right\}$$

$$\left.\begin{aligned} \lambda &= h \cos \phi \\ \mu &= h \cosh n(p+\phi) \end{aligned}\right\}$$

and these two systems are the solutions of the two distinct differential equations

$$\frac{d\lambda}{\sqrt{\lambda^2 - h^2}} = n \frac{d\mu}{\sqrt{h^2 - \mu^2}},$$

$$n \frac{d\lambda}{\sqrt{h^2 - \lambda^2}} = \frac{d\mu}{\sqrt{\mu^2 - h^2}}.$$

If, now, we consider for a moment these two differential equations, we see that the first belongs to the trajectory which intersects the confocal ellipses at an angle α (where $n = \tan \alpha$), while the other belongs to the trajectory which intersects the confocals at an angle $\left(\frac{\pi}{2} - \alpha\right)$, since

$$\frac{1}{n} = \tan \left(\frac{\pi}{2} - \alpha\right).$$

But, since the confocal hyperbolas intersect the ellipses orthogonally, it follows at once that the second differential equation belongs to the trajectory which intersects the confocal hyperbolas at an angle $(\pi - \alpha)$, in both cases measuring the angle in the same sense; hence, the solution

$$\mathrm{M} = \frac{h\mu}{\lambda},$$

which leads to the system

$$\begin{aligned} \lambda &= h \cosh n(p+\phi) \\ \mu &= h \cos \phi \end{aligned}$$

is relevant, while the value

$$\mathrm{M} = \frac{h\lambda}{\mu},$$

which furnishes the other system

$$\begin{aligned} \lambda &= h \cos \phi \\ \mu &= h \cosh n(p+\phi) \end{aligned}$$

is irrelevant. We conclude, therefore, that, of the two solutions to which Mainardi's result is really equivalent, only one is relevant; the other being wholly extraneous, as belonging to the oblique trajectory of the orthogonal system of confocal hyperbolas;* and, it is easy to discriminate which of the two solutions given by the quadratic

$$(x^2+y^2+h^2)\mathrm{M} = x\,(\mathrm{M}^2+h^2)$$

leads to the relevant solution; for we have seen that the solution in point is furnished by

$$\mathrm{M} = \frac{h\mu}{\lambda};$$

now it is evident geometrically that

$$\lambda > h > \mu,$$

which shews at once that

$$\frac{h\mu}{\lambda} < \frac{h\lambda}{\mu};$$

it follows, therefore, that the smaller of the two roots of the quadratic in M is the proper value. We come to the conclusion, therefore, that in Mainardi's system

$$-2n\tan^{-1}\sqrt{\frac{h^2}{x\mathrm{M}}-1}+\log\frac{1-\sqrt{1-\dfrac{\mathrm{M}}{x}}}{1+\sqrt{1-\dfrac{\mathrm{M}}{x}}}=\mathrm{C},$$

$$(x^2+y^2+h^2)\,\mathrm{M} = x(\mathrm{M}^2+h^2),$$

the smaller root of the quadratic in M gives the oblique trajectory of the system of confocal ellipses, while the greater root furnishes the oblique trajectory of the system of confocal hyperbolas. I am not aware that the real character of the two solutions to which Mainardi's result is equivalent has been before distinguished as above.

Lastly, it is sufficiently obvious that the values of λ, μ given by either of the above systems may be geometrically represented by a construction closely analogous to what is given in my former paper mentioned at the beginning of this memoir.

10*th December* 1887.

* Instances of a single solution resolving itself into two, are by no means rare; for example, in the case of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

this equation is really equivalent to the two

$$by = -(hx+f)+\sqrt{(h^2-ab)x^2+2(hf-bg)x+(f^2-bc)}$$

$$by = -(hx+f)-\sqrt{(h^2-ab)x^2+2(hf-bg)x+(f^2-bc)}$$

Bnt the present case is distinguishable from the case of the conic, inasmuch as we have here one of the solutions irrelevant, while, in the case of the conic, both the solutions are relevant, the compound solution being reproduced by multiplying together the resolved solutions.